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# Pure-spinor superstrings in $d = 2, 4, 6$

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Abstract: We continue the study of the $d = 2, 4, 6$ pure-spinor superstring models introduced in [1]. By explicitly solving the pure-spinor constraint we show that these theories have vanishing central charge and work out the (covariant) current algebra for the Lorentz currents. We argue that these super-Poincaré covariant models may be thought of as compactifications of the superstring on $CY_{4,3,2}$, and take some steps toward making this precise by constructing a map to the RNS superstring variables. We also discuss the relation to the so called hybrid superstrings, which describe the same type of compactifications.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory.

## Contents



## 1. Introduction

A couple of years ago Berkovits proposed a new approach to the quantisation of the ten-dimensional superstring [2] (see also [3] for variants of this idea). This so called pure-spinor superstring has the virtue that it has manifest ten-dimensional super-Poincaré covariance. For a review see [4].

A natural question to ask is if there are pure-spinor superstrings in lower dimensions, for instance arising via compactifications. In [1] we introduced pure-spinor superstring theories[1] in $d = 2, 4, 6$ by mimicking Berkovits' construction in $d = 10$ [2]. As in $d = 10$, the "ghost" sector of these models involve constrained ("pure") bosonic spinors. The (quadratic) constraints on these spinors were first written down in [4] and were discussed in [1].

In this paper we show that the constraints on the $\lambda$'s are such that they imply that the world-sheet conformal field theories for the $d = 2, 4, 6$ models have $c = 0$ (vanishing total central charge) and $k = 1$ (total matter+ghost Lorentz current algebra has level one). The approach we follow to obtain these results is the same as the one originally followed in $d = 10$ [2, 5]. This is perhaps not the most elegant method since it temporarily breaks

[1] Pure-spinor superparticle models in $d = 4, 6$ were introduced earlier in section 2.6 of [4] and correspond to the particle limit of the models in [1].

manifest covariance,[2] but on the other hand the method is straightforward and leads to expressions which can be compared with corresponding ones in the Ramond-Neveu-Schwarz (RNS) superstring. For an alternative covariant approach see [6, 7].

In view of the well-known fact that the superstring is only consistent in $d = 10$ the above results might seem surprising. The existence of $d = 4, 6$ pure-spinor *superparticle* models is not surprising given that covariant Brink-Schwarz superparticle models [8] exist in these dimensions. The $d = 4, 6$ Brink-Schwarz superparticle models can be quantised in the light-cone gauge and it should be possible to relate them to the pure-spinor models (this appears to be slightly subtle though; see section 2.6 in [4] for a discussion). On the other hand, if one tries to quantise the corresponding $d = 4, 6$ Green-Schwarz (GS) superstring theories [9] in the light-cone gauge one finds that this leads to an inconsistency since the Lorentz algebra does not close (see e.g. [10] for a discussion). For the pure-spinor models, in contrast to the situation in the light-cone GS superstring, Lorentz covariance is not broken by quantum effects. However, we do *not* claim that the pure-spinor models are consistent critical superstring theories[3] in dimensions $d < 10$. Rather we will argue that the pure-spinor models should be thought of as the non-compact piece of compactifications of the ten-dimensional superstring on 4,3,2 (complex) dimensional Calabi-Yau (CY) manifolds from $d = 10$ to $d = 2, 4, 6$. The inconsistency of the lower-dimensional theory without the CY piece, which in the RNS superstring appears as $c \neq 0$ and in the light-cone GS superstring as a breakdown of Lorentz covariance, for the pure-spinor models should appear in another sector of the theory (exactly where the inconsistency appears we do not yet fully understand).

An important question to understand is the relation between the $d = 2, 4, 6$ pure-spinor models and the compactified RNS superstring. We will argue that the relation is analogous to the one in the uncompactified case [5] (see also [12]). More precisely, we will show that after a change of variables from the RNS variables plus the addition of a certain number of $c = 0$ "topological quartets", the field content and stress tensor precisely matches that of the pure-spinor models plus a decoupled sector describing the CFT of the compactification manifold.

With the interpretation of the pure-spinor models as compactified theories with manifest Lorentz covariance in the non-compact directions, the question arises how these models are related to the so called hybrid superstrings [13, 14] which describe the same type of compactifications. Roughly, we find that after the addition of a certain $c = 0$ piece, the stress tensor in the hybrid superstring agrees with the stress tensor of the corresponding pure-spinor superstring written in terms of Lorentz covariant variables. In comparison to the hybrid superstrings, an advantageous feature of the pure-spinor models is that they seemingly circumvent many of the problematic aspects associated with the negative-energy chiral scalars present in the hybrid models. However, it should be stressed that the pure-spinor models need to be developed further before they can be considered as a viable alternatives to the hybrid models. In particular, the BRST operator, vertex operators and scattering amplitudes need to be better understood. Another application of the $d = 2, 4, 6$

[2] As in $d = 10$, in $d = 2n$ we solve the pure-spinor constraint in terms of free fields by temporarily breaking the manifest SO($2n$) (Wick rotated) Lorentz invariance to U($n$) $\simeq$ SU($n$)$\times$ U(1).

[3] For a recent discussion of covariant non-critical superstrings see [11].



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pure-spinor conformal field theories presented in this paper is as toy models for the more involved $d = 10$ pure-spinor conformal field theory.

This paper is organised as follows. In the next section the $d = 4$ pure-spinor model is discussed. In section 3 the same analysis is carried out for the $d = 6$ model, and in section 4 the $d = 2$ model is briefly discussed. Then in section 5 the relation to the RNS superstring is discussed followed in section 6 by a discussion of the relation to the hybrid superstrings. A discussion of open problems and some applications are presented in section 7. Finally, in the appendix some technical results are collected.

## 2. The $d = 4$ pure-spinor superstring

In this section we consider the $d = 4$, $\mathcal{N} = 1$ pure-spinor superstring [1]. For simplicity we consider the type-II string in a flat supergravity background and write only the left-moving worldsheet fields explicitly. The left-moving (holomorphic) "matter" worldsheet fields are $(x^m, \theta^\alpha, p_\alpha)$, where $\theta^\alpha$ is a four-component Dirac spinor,[4] and $p_\alpha$ is its conjugate momentum ($\alpha = 1, \ldots, 4$). The Dirac spinor $\theta^\alpha$ can be decomposed into a Weyl spinor, $\theta^a$ ($a = 1, 2$), and an anti-Weyl spinor, $\bar\theta^{\dot a}$ ($\dot a = 1, 2$). Similarly, $p_\alpha$ can be decomposed into $p_a$, and $\bar p_{\dot a}$. The free fields $x^m$, $\theta^\alpha$ and $p_\alpha$ have the standard OPEs (in units where $\alpha' = 2$)

$$x^m(y, \bar y)\, x^n(z, \bar z) \sim -\eta^{mn} \log |y - z|^2 \,, \qquad p_\alpha(y)\, \theta^\beta(z) \sim \frac{\delta_\alpha^\beta}{y - z} \,. \tag{2.1}$$

In the Weyl basis one has

$$p_a(y)\, \theta^b(z) \sim \frac{\delta_a^b}{y - z} \,, \qquad \bar p_{\dot a}(y)\, \bar\theta^{\dot b}(z) \sim \frac{\delta_{\dot a}^{\dot b}}{y - z} \,. \tag{2.2}$$

As in the $d = 10$ pure-spinor formalism, the world-sheet ghost fields involve a Grassmann-even spinor, $\lambda^\alpha$ (and its conjugate momentum). The bosonic (Dirac) spinor $\lambda^\alpha$ is assumed to satisfy the "pure spinor" condition

$$\lambda \Gamma^m \lambda = 0 \,. \tag{2.3}$$

Here $\Gamma^m_{\alpha\beta}$ are the (symmetric) $4{\times}4$ gamma matrices (we do not explicitly write the charge conjugation matrix used to lower indexes). In the Weyl basis, the above condition can be written as $\lambda^a \bar\lambda^{\dot b} = 0$ (more details are given below).

The conjugate momentum to $\lambda^\alpha$ will be denoted $w_\alpha$. As in the $d = 10$ pure-spinor superstring, because of the pure-spinor constraint, $w_\alpha$ is only defined up to the gauge invariance: $w_\alpha \to w_\alpha + \Lambda_m (\Gamma^m \lambda)_\alpha$.

As in $d = 10$, one can construct the Lorentz covariant quantities

$$N^{mn} = \frac{1}{2}\, w \Gamma^{mn} \lambda \,, \qquad \partial h = \frac{1}{2}\, w \lambda \,. \tag{2.4}$$

[4] The slightly unfortunate Dirac spinor index notation is chosen to make the $d = 4$ formulæ correspond as closely as possible to the $d = 10$ ones.



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To calculate the OPEs between involving $N^{mn}$ and $\partial h$ one can either (temporarily) explicitly solve the pure-spinor constraint and calculate the OPEs by expressing $N^{mn}$ and $\partial h$ in terms of the unconstrained variables, verifying Lorentz covariance at the end, or one can use the covariant method introduced in [6]. Here we will follow the former approach, which has the advantage that it also gives us information about the zero-mode saturation rule and a possible relation to the RNS and hybrid superstring (see later sections). The covariant approach is discussed in [7].

One can solve the pure-spinor constraint (2.3) in terms of free fields by temporarily breaking the SO(4) (Wick rotated) Lorentz group to U(2) $\simeq$ SU(2)$\times$ U(1). Under this subgroup $\lambda^\alpha$ decomposes into (see appendix A for further details) $(\lambda^+, \lambda^a, \lambda_{ab})$, where $\lambda_{ab} = -\lambda_{ba}$. Here $a, b = 1, 2$. In this U(2) basis the pure-spinor condition becomes

$$\lambda^+\lambda^a = 0\,, \qquad \lambda_{ab}\lambda^b = 0\,. \tag{2.5}$$

We have chosen to write the $d = 4$ pure spinor condition in a way which is closely analogous to the $d = 10$ case. The $d = 4$ pure-spinor condition can also be conveniently written as $\lambda^a\bar{\lambda}^{\dot{a}} = 0$ where we have introduced the notation $\bar{\lambda}^{\dot{a}} = \{\lambda^+, \frac{1}{2}\epsilon^{ab}\lambda_{ab}\}$. In our conventions, as the notation suggests, $\lambda^a$ is a Weyl spinor whereas $\bar{\lambda}^{\dot{a}}$ is an anti-Weyl spinor.

The explicit solution to the constraints (2.5) is

$$\{\lambda^a = 0\} \cup \{\bar{\lambda}^{\dot{a}} = 0\}\,. \tag{2.6}$$

It is important to note that the solution has two "patches", $\lambda^a = 0$ and $\bar{\lambda}^{\dot{a}} = 0$. Both patches are part of the solution and should be taken into account (see [1] for a discussion). Below we are mostly concerned with quantities which do not depend on where in the space of solutions (2.6) one is and we therefore work in one of the two patches, $\lambda^a = 0$.

The pure-spinor constraint (2.3) eliminates two components from $\lambda^\alpha$. The remaining independent (unconstrained) components of $\lambda^\alpha$ have canonical free-field OPEs with the corresponding components of the conjugate momentum. The components of the conjugate momentum corresponding to the two eliminated components of $\lambda^\alpha$ can be gauged to zero. For instance the gauge $w_a = 0$ can be chosen in the $\lambda^a = 0$ patch.

In the U(2) basis the Lorentz covariant quantities (2.4) can be written

$$\begin{aligned} N^{ab} &= \frac{1}{2}w^{ab}\lambda^+\,, \qquad N_{ab} = -\frac{1}{2}w_+\lambda_{ab} - \frac{1}{2}\frac{\partial\lambda_{ab}}{\lambda^+}\,, \\ N^a{}_b &= -\frac{1}{4}\delta^a_b[w_+\lambda^+ + \frac{\partial\lambda^+}{\lambda^+} - \frac{1}{2}w^{ab}\lambda_{ab}]\,, \end{aligned} \tag{2.7}$$

as well as

$$\partial h = \frac{1}{2}w_+\lambda^+ - \frac{1}{2}\frac{\partial\lambda^+}{\lambda^+} + \frac{1}{4}w^{ab}\lambda_{ab}\,. \tag{2.8}$$

Here the terms with derivatives are related to normal-ordering ambiguities. The normal-ordering ambiguities are chosen so that $\partial h$ and $N^{mn}$ have no OPEs with each other and SO(4) Lorentz covariance is preserved (see below).

The above ghost Lorentz currents (2.7) satisfy the OPEs

$$\begin{aligned}
N^a{}_b(y)N^c{}_d(z) &\sim \frac{k}{4}\frac{\delta^a_d\delta^c_b}{(y-z)^2}+\frac{1}{2}\frac{\delta^a_d N^c{}_b-\delta^c_b N^a{}_d}{y-z}\\
N^{ab}(y)N^c{}_d(z) &\sim -\frac{\delta^{[a}_d N^{b]c}}{y-z}\\
N^{ab}(y)N_{cd}(z) &\sim -\frac{k}{2}\frac{\delta^{ab}_{cd}}{(y-z)^2}+2\frac{\delta^{[a}_{[c}N^{b]}{}_{d]}}{y-z}\\
N_{ab}(y)N^c{}_d(z) &\sim \frac{\delta^c_{[a}N_{b]d}}{y-z}
\end{aligned} \tag{2.9}$$

where $k=0$ and, as usual, $\delta^{ab}_{cd}=\frac{1}{2}(\delta^a_c\delta^b_d-\delta^a_d\delta^b_c)$. In manifestly SO(4) covariant notation one finds that the OPEs involving $N^{mn}$ and $\lambda^\alpha$ take the form

$$\begin{aligned}
N^{mn}(y)\lambda^\alpha(z) &\sim \frac{1}{2}\frac{1}{y-z}\left(\Gamma^{mn}\lambda\right)^\alpha(z)\\
N^{pq}(y)N^{mn}(z) &\sim \frac{\eta^{pm}N^{qn}(z)-\eta^{qm}N^{pn}(z)-(m\leftrightarrow n)}{y-z}\,.
\end{aligned} \tag{2.10}$$

The second equation in (2.10) shows that $N^{mn}$ satisfies an SO(4) current algebra with level $k=0$. Furthermore, $h$ has no singular OPEs with $N^{mn}$ and satisfies

$$h(y)h(z)\sim-\log(y-z)\,,\qquad \partial h(y)\lambda(z)\sim\frac{1}{2}\frac{1}{y-z}\lambda(z)\,. \tag{2.11}$$

Notice that the above results are completely analogous to the ones in $d=10$ (see e.g. section 2 of ref. [15] for a discussion of the $d=10$ case using closely related notation and conventions).

Above we found that $N^{mn}$ satisfies an SO(4) current algebra with level $k=0$. In comparison, the OPEs involving the $(p,\theta)$ Lorentz currents, $M^{mn}=-\frac{1}{2}p\Gamma^{mn}\theta$, take the form

$$\begin{aligned}
M^{mn}(y)\theta^\alpha(z) &\sim \frac{1}{2}\frac{1}{y-z}(\Gamma^{mn}\theta)^\alpha(z)\\
M^{pq}(y)M^{mn}(z) &\sim \frac{\eta^{pm}M^{qn}(z)-\eta^{qm}M^{pn}(z)-(m\leftrightarrow n)}{y-z}+\frac{\eta^{pn}\eta^{qm}-\eta^{pm}\eta^{qn}}{(y-z)^2}\,.
\end{aligned} \tag{2.12}$$

Thus the $M^{mn}$'s form an SO(4) current algebra with level $k=1$. The total Lorentz current, $L_{mn}=M^{mn}+N^{mn}$, thus satisfies the OPE

$$L^{pq}(y)L^{mn}(z)\sim\frac{\eta^{pm}L^{qn}(z)-\eta^{qm}L^{pn}(z)-(m\leftrightarrow n)}{y-z}+\frac{\eta^{pn}\eta^{qm}-\eta^{pm}\eta^{qn}}{(y-z)^2}\,, \tag{2.13}$$

forming a current algebra with level $k=1$ (just as in the $d=10$ case).

The "matter" part of the stress tensor for the $d=4$ pure-spinor superstring can be written [1]

$$T_{\mathrm{mat}}=-\frac{1}{2}\partial x_m\partial x^m-p_\alpha\partial\theta^\alpha\,. \tag{2.14}$$

From this expression one sees that the $x^m$ CFT has central charge $c = 4$, while the $(p, \theta)$ CFT has central charge $c = -8$. In order for the total central charge to vanish, the ghost CFT has to have $c = 4$.

Using the covariant fields $N^{mn}$ and $\partial h$, the $d = 4$ ghost stress tensor can be written in a manifestly Lorentz invariant way as

$$T_{N,\partial h} = -\frac{1}{8} N_{mn} N^{mn} - \frac{1}{2} (\partial h)^2 + \frac{1}{2} \partial^2 h \,. \tag{2.15}$$

The terms in the above expression are all that are allowed by Lorentz invariance and the conformal weight of the stress tensor. The requirement that $N^{mn}$ should have conformal weight one as should $\partial h$ (except for the background charge) and that $\lambda$ should have conformal weight zero puts restrictions on the coefficients.

By using (2.7) and (2.8) together with various normal ordering rearrangements, one can check that the stress tensor (2.15) reduces to

$$T_{w,\lambda} = w_+ \partial \lambda^+ + \frac{1}{2} w^{ab} \partial \lambda_{ab} \,, \tag{2.16}$$

written in terms of the U(2) variables. Thus the ghost sector comprise two $\beta\gamma$ systems of weight one. The central charge of the ghost stress tensor is therefore $c = 4$ and hence the total central charge vanishes. This result can also be seen directly from (2.15). The first piece involves the ghost Lorentz currents, $N^{mn}$, and is a Sugawara construction for an SO(4) WZNW model with level $k = 0$. Indeed, recalling that the dual Coxeter number of SO($2n$) is $g^\vee = 2n - 2$, we find[5] $2(g^\vee + k) = 4$. Using standard formulæ, the central charge for an SO($2n$) Lorentz current algebra with level $k$ is

$$c = \frac{k \dim \mathrm{SO}(2n)}{k + g^\vee} \,, \tag{2.17}$$

which vanishes when $k = 0$. In (2.15) the pieces involving $\partial h$ refer to a Coulomb gas with background charge $Q = 1$, and consequently central charge $c = 1 + 3Q^2 = 4$. As above, the total ghost central charge is $c = 4$.

As in $d = 10$ it turns out that one can also write the $(p, \theta)$ part of the stress tensor in a form similar to what was done above for the ghost part. It is unclear if this is just a curiosity or whether it can be useful. The analogue of $\partial h$ in the $(p, \theta)$ sector is $\partial g = \frac{1}{2} p_\alpha \theta^\alpha$. One can also introduce $\partial \hat{g} = \frac{1}{2} p \Gamma_5 \theta$. The two scalars $g$, $\hat{g}$ have no singularities with $M^{mn}$ and satisfy

$$g(y)\, g(z) \sim \log(y - z) \,, \qquad \hat{g}(y)\, \hat{g}(z) \sim \log(y - z) \,, \qquad g(y)\, \hat{g}(z) \sim 0 \,. \tag{2.18}$$

In terms of $M_{mn}$, $\partial g$ and $\partial \hat{g}$ one finds that $T_{p\theta} = -p_\alpha \partial \theta^\alpha$ can be rewritten as

$$T_{M,\partial g} = -\frac{1}{12} M_{mn} M^{mn} + \frac{1}{2} (\partial \hat{g})^2 + \frac{1}{2} (\partial g)^2 - \partial^2 g \,. \tag{2.19}$$

The central charge obtained from (2.19) is $c = 2 + 1 + 1 - 3 \cdot 2^2 = -8$ as it should be.

[5]The prefactor in front of $N_{mn} N^{mn}$ in (2.15) is $-\frac{1}{4(k+g^\vee)}$. To obtain the more conventional $+\frac{1}{2(k+g^\vee)}$ one would have to rescale the currents $N^{mn}$.

In contrast to the $d = 10$ case there is a structural difference between (2.15) and (2.19) in that in the latter $\hat{g}$ appears but in the former no corresponding $\hat{h}$ appears. As we will see in the next section a similar result also holds for the $d = 6$ model (see this section for a longer discussion).

The zero-mode saturation rule for the above model can be obtained by noting that $T_{p,\theta}$ comprise four $bc$-type systems and $T_{w,\lambda}$ comprise two $\beta\gamma$-type systems, all of weight one. By using standard methods [16] one finds that the saturation rule is

$$\langle 0|\epsilon_{\dot{a}\dot{b}}\bar{\theta}^{\dot{a}}\bar{\theta}^{\dot{b}}[\theta^{+}\theta_{ab}][\delta(\lambda^{+})\delta(\lambda_{ab})]|0\rangle \neq 0\,, \tag{2.20}$$

which is equivalent to

$$\langle 0|\theta^2|\Omega\rangle \neq 0\,, \tag{2.21}$$

where (as in $d = 10$) $|\Omega\rangle = \prod_{A=1}^{2} Y^A|0\rangle$ with $Y^A = C^A_\alpha\theta^\alpha\delta(C^A_\alpha\lambda^\alpha)$ and $C^A_\alpha$ are certain constant bosonic spinors. The two operators $Y^A$ each carry $(\lambda, \theta)$ charge $(-1, +1)$ and one can check that the above saturation rule is consistent with the background charges of $\partial h$ and $\partial g$.

The next step in the analysis of the above model would be to construct a BRST operator and analyse vertex operators and scattering amplitudes. These questions were touched upon in [1]. The naive BRST operator (based on the $d = 10$ expression)

$$Q = \oint \lambda^\alpha d_\alpha\,, \tag{2.22}$$

has *off-shell* $\mathcal{N} = 1$ super-Yang-Mills as its massless cohomology at ghost number 1 [4, 1]. The fact that the massless cohomology is off-shell SYM seems to indicate that the above BRST operator can not be the full story. Vertex operators corresponding to the above BRST operator were briefly discussed in [1]. The fact that the unintegrated vertex operator for the massless states, $U$, has ghost number 1 together with the form of the above saturation rule seems to require that the three unintegrated vertex operators in tree-level scattering amplitudes should have total ghost number zero. This seems to indicate that the same construction as in $d = 10$ [2, 17] can not work here without modification. We hope to return to these questions in the future.

To summarise: in this section we analysed the conformal field theory for the $d = 4$ pure-spinor superstring and showed that it has vanishing central charge and is such that the Lorentz current algebra has level one. We also obtained the zero-mode saturation rule. Many open problems remain, some of which were mentioned above.

## 3. The $d = 6$ pure-spinor superstring

In this section the same analysis that was carried out in $d = 4$ in the previous section will be performed for the $d = 6$ case with minimal supersymmetry. As in $d = 4$ we work in a flat type-II supergravity background and only display the left-moving sector. The left-moving (holomorphic) "matter" worldsheet fields in $d$=6 with $\mathcal{N} = (1, 0)$ supersymmetry are [1] $(x^m, \theta^\alpha_I, p^I_\alpha)$, where $\theta^\alpha_I$ is a doublet ($I = 1, 2$) of four-component Weyl spinors, and $p^I_\alpha$ are their conjugate momenta. (Note that in this section $\alpha, \beta, \ldots$ denote Weyl indexes, and not Dirac indexes as in $d = 4$.)



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By analogy with the $d = 10$ and $d = 4$ cases we take the world-sheet ghost fields to involve a doublet of Grassmann-even Weyl spinors, $\lambda_I^\alpha$ ($I = 1, 2$), (and their conjugate momenta) and impose the pure-spinor condition [4, 1]

$$\epsilon^{IJ}\lambda_I\gamma^m\lambda_J = 0\,. \tag{3.1}$$

Here $\gamma^m$ are the (antisymmetric) 4×4 off-diagonal blocks ("Pauli matrices") in the Weyl representation of the 8×8 six-dimensional gamma matrices $\Gamma^m$. Note that the above condition (3.1) is *not* a conventional pure-spinor condition in the sense of Cartan (which is solved by a Weyl spinor). However, as confusion is unlikely to arise we refer to (3.1) as a pure-spinor condition throughout.

The free fields $x^m$, $\theta_I^\alpha$ and $p_\alpha^I$ have the standard OPEs (in units where $\alpha' = 2$),

$$x^m(y,\bar{y})x^n(z,\bar{z}) \sim -\eta^{mn}\log|y-z|^2\,, \qquad p_\alpha^I(y)\theta_J^\beta(z) \sim \frac{\delta_J^I\delta_\alpha^\beta}{y-z}\,. \tag{3.2}$$

As in $d = 4, 10$ we can solve the pure-spinor constraint in terms of free fields by temporarily breaking the manifest SO(6) (Wick rotated) Lorentz invariance to U(3) $\simeq$ SU(3)× U(1). Under this subgroup $\lambda_I^\alpha$ decomposes into (see appendix A for further details) $(\lambda_I^+, \lambda_I^a)$, where $a, b = 1, \ldots, 3$.

In the U(3)-basis the pure-spinor condition becomes

$$\epsilon^{IJ}\lambda_I^+\lambda_J^a = 0\,, \qquad \epsilon_{abc}\epsilon^{IJ}\lambda_I^a\lambda_J^b = 0\,. \tag{3.3}$$

An explicit solution to these constraints is

$$\lambda_2^a = \frac{\lambda_2^+}{\lambda_1^+}\lambda_1^a\,, \tag{3.4}$$

which can also be written as $\lambda_2^\alpha = e^{-v}\lambda_1^\alpha$, where $v = \ln(\lambda_1^+/\lambda_2^+)$. Thus, we see that the pure-spinor condition (3.1) eliminates 3 components from $\lambda_I^\alpha$. The remaining five independent (unconstrained) components of $\lambda_I^\alpha$ have canonical free-field OPEs with the corresponding components of the conjugate momenta, $w_\alpha^I$. Using the gauge symmetry induced by the pure-spinor constraint, $\delta w_\alpha^I = \epsilon^{IJ}\Lambda_m(\gamma^m\lambda_J)_\alpha$ the components of the conjugate momentum corresponding to the three constrained components of $\lambda_I^\alpha$ can be gauged to zero, e.g. $w_a^2 = 0$.

As in $d = 4, 10$, one can construct the SO(6) Lorentz covariant quantities

$$N^{mn} = \frac{1}{2}w^I\gamma^{mn}\lambda_I\,, \qquad \partial h = \frac{1}{2}w^I\lambda_I\,. \tag{3.5}$$

In the U(3) basis one has:

$$\begin{aligned} N^{ab} &= \frac{1}{2}\epsilon^{abc}w_c^1\lambda_1^+\,, \\ N_{ab} &= -\frac{1}{2}\epsilon_{abc}w_+^1\lambda_1^c - \frac{1}{2}\epsilon_{abc}(w_+^2\lambda_2^+)\frac{\lambda_1^c}{\lambda_1^+} + \frac{a_1}{2}\epsilon_{abc}\frac{\partial\lambda_1^+}{(\lambda_1^+)^2}\lambda_1^c + \frac{-a_1-\frac{1}{2}}{2}\epsilon_{abc}\frac{\partial\lambda_1^+}{\lambda_1^+\lambda_2^+}\lambda_1^c\,, \\ N^a{}_b &= -\frac{1}{2}w_b^1\lambda_1^a - \frac{1}{4}\delta_b^a\left[w_+^1\lambda_1^+ + w_+^2\lambda_2^+ - w_c^1\lambda_1^c - a_1\frac{\partial\lambda_1^+}{\lambda_1^+} + \left(a_1+\frac{1}{2}\right)\frac{\partial\lambda_2^+}{\lambda_2^+}\right], \end{aligned} \tag{3.6}$$

and

$$\partial h = \frac{1}{2}\left[w_+^1\lambda_1^+ + w_+^2\lambda_2^+ + w_c^1\lambda_1^c + (2-a_1)\frac{\partial\lambda_1^+}{\lambda_1^+} + \left(a_1 - \frac{3}{2}\right)\frac{\partial\lambda_2^+}{\lambda_2^+}\right]. \tag{3.7}$$

It will also be convenient to introduce

$$\begin{aligned}\partial u &= \frac{1}{2}\left[w_+\lambda_1^+ + w_c^1\lambda_1^c - w_+^2\lambda_2^+ + (a_1+1)\frac{\partial\lambda_1^+}{\lambda_1^+} + \left(a_1 - \frac{3}{2}\right)\frac{\partial\lambda_2^+}{\lambda_2^+}\right],\\ \partial v &= \frac{\partial\lambda_1^+}{\lambda_1^+} - \frac{\partial\lambda_2^+}{\lambda_2^+}.\end{aligned} \tag{3.8}$$

In the above expressions, the terms with derivatives are related to normal-ordering ambiguities and $a_1$ is an arbitrary constant. We have indicated the normal-ordering prescription by parentheses. The normal-ordering terms are restricted by the requirement that the OPEs be Lorentz covariant, and the fact that they can be chosen such that this is true is a non-trivial result.

The above Lorentz currents for the ghosts satisfy, in the U(3) basis (3.6), the same OPEs as in (2.9) but with $k = -1$. In manifestly SO(6) covariant notation the OPEs involving $N^{mn}$ and $\lambda_I^\alpha$ take the form

$$\begin{aligned}N^{mn}(y)\lambda_I^\alpha(z) &\sim \frac{1}{2}\,\frac{1}{y-z}(\gamma^{mn})^\alpha{}_\beta\,\lambda_I^\beta(z)\,,\\ N^{pq}(y)N^{mn}(z) &\sim \frac{\eta^{pm}N^{qn}(z) - \eta^{qm}N^{pn}(z) - (m\leftrightarrow n)}{y-z} - \frac{\eta^{pn}\eta^{qm} - \eta^{pm}\eta^{qn}}{(y-z)^2}.\end{aligned} \tag{3.9}$$

Thus, the $N^{mn}$'s form an SO(6) current algebra with level $k = -1$. As in $d = 4, 10$ $h$ has no singular OPEs with $N^{mn}$, and satisfies

$$h(y)h(z) \sim -\log(y-z)\,, \qquad \partial h(y)\lambda_I^\alpha(z) \sim \frac{1}{2}\,\frac{1}{y-z}\lambda_I^\alpha(z)\,. \tag{3.10}$$

In addition, the worldsheet fields $\partial u$ and $\partial v$ have no singularities with $N^{mn}$ or $\partial h$ or with themselves and satisfy

$$\partial u(y)\,\partial v(z) \sim \frac{1}{(y-z)^2}\,. \tag{3.11}$$

Furthermore,

$$\partial u(y)\,\lambda_1^\alpha(z) \sim \frac{1}{2}\frac{1}{(y-z)}\lambda_1^\alpha\,, \qquad \partial u(y)\,\lambda_2^\alpha(z) \sim -\frac{1}{2}\frac{1}{(y-z)}\lambda_2^\alpha\,. \tag{3.12}$$

In comparison to the results in (3.9) the OPEs involving the $(p,\theta)$ Lorentz currents, $M^{mn} = -\frac{1}{2}p^I\gamma^{mn}\theta_I$, take the form

$$\begin{aligned}M^{mn}(y)\theta_I^\alpha(z) &\sim \frac{1}{2}\,\frac{1}{y-z}(\gamma^{mn})^\alpha{}_\beta\,\theta_I^\beta(z)\\ M^{pq}(y)M^{mn}(z) &\sim \frac{\eta^{pm}M^{qn}(z) - \eta^{qm}M^{pn}(z) - (m\leftrightarrow n)}{y-z} + 2\frac{\eta^{pn}\eta^{qm} - \eta^{pm}\eta^{qn}}{(y-z)^2}.\end{aligned} \tag{3.13}$$

Thus the $M^{mn}$'s form an SO(6) current algebra at level $k=2$. The total Lorentz current $L_{mn} = M^{mn} + N^{mn}$ satisfies the OPE

$$L^{pq}(y)L^{mn}(z) \sim \frac{\eta^{pm}L^{qn}(z) - \eta^{qm}L^{pn}(z) - (m \leftrightarrow n)}{y-z} + \frac{\eta^{pn}\eta^{qm} - \eta^{pm}\eta^{qn}}{(y-z)^2}\,, \tag{3.14}$$

and thus forms a current algebra with level $k=1$.

The "matter" part of the stress tensor for the $d=6$ pure-spinor superstring can be written

$$T_{\text{mat}} = -\frac{1}{2}\partial x_m \partial x^m - p^I_\alpha \partial\theta^\alpha_I\,. \tag{3.15}$$

From this expression one sees that the $x^m$ CFT has central charge $c=6$, while the $(p,\theta)$ CFT has central charge $c=-16$. In order for the total central charge to vanish, the ghost CFT has to have $c=10$.

Using the Lorentz covariant fields $N^{mn}$, $\partial h$, $\partial u$ and $\partial v$ the ghost stress tensor can be written in a manifestly Lorentz invariant way as

$$T_{N,\partial h} = -\frac{1}{12}N_{mn}N^{mn} - \frac{1}{2}(\partial h)^2 + \partial^2 h + \partial u \partial v - \partial^2 v\,. \tag{3.16}$$

After using various normal-ordering rearrangements it can be shown that the above stress tensor reduces to

$$T_{w,\lambda} = \left[(w^2_+\lambda^+_2) + \frac{1}{2}\frac{\partial\lambda^+_2}{\lambda^+_2}\right]\frac{\partial\lambda^+_2}{\lambda^+_2} + w^1_+\partial\lambda^+_1 + w^1_c\partial\lambda^c_1\,. \tag{3.17}$$

In this form it is easy to verify the conformal dimensions of $N^{mn}$ and $\lambda^\alpha_I$ and check that $c=10$ so that the total central charge vanishes.

We note in passing that there is in fact a slight ambiguity in writing (3.16) since if one uses $-\frac{1}{2}\partial^2 v$ instead of $-\partial^2 v$ one is lead to the same expression as in (3.17) but with the roles of $\lambda^+_1$ and $\lambda^+_2$ interchanged.

The central charge can of course also be obtained from (3.16). The first piece is a Sugawara construction for a SO(6) WZNW model with level $k=-1$ and consequently central charge $c=-5$. The piece involving $\partial h$ has $c = 1 + 3\cdot 2^2 = 13$. Finally, the $\partial u$, $\partial v$ piece has $c=2$ for a total of $c=10$.

One can also rewrite the $(p,\theta)$ part of the stress tensor in a form similar to what was done above for the ghost part. Besides $M^{mn} = -\frac{1}{2}p_I\gamma^{mn}\theta^I$ it is also convenient to introduce $\partial g^I{}_J = p^I\theta_J$, which can be decomposed into:

$$\partial g = \frac{1}{2\sqrt{2}}\,\partial g^I{}_I\,, \qquad R^I{}_J = \partial g^I{}_J - \frac{1}{2}\,\delta^I_J\,\partial g\,. \tag{3.18}$$

The fields $\partial g^I{}_J$ have no singularities with $M^{mn}$ and satisfy

$$\partial g^I{}_J(y)\,\partial g^K{}_L(z) \sim 4\frac{\delta^I_L\delta^K_J}{(y-z)^2} + \frac{\delta^I_L\partial g^K{}_J - \delta^I_L\partial g^K{}_J}{y-z}\,. \tag{3.19}$$

From this result it follows that $\partial g(y)\partial g(z) \sim \log(z-y)$ and that $R^I{}_J$ form an SU(2) (or Sp(2)) current algebra with level $k=4$ (which, as opposed to the $M^{mn}$ and $N^{mn}$ current algebras, is conventionally normalised).



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One can show that $T_{p\theta} = -p^I_\alpha \partial\theta^\alpha_I$ is equal to

$$T_{M,\partial g} = -\frac{1}{24} M_{mn} M^{mn} + \frac{1}{12} R^I{}_J R^J{}_I + \frac{1}{2} (\partial g)^2 - \sqrt{2}\, \partial^2 g \,. \tag{3.20}$$

In this form, the total central charge is calculated to be $c = 5 + 2 + 1 - 3 \cdot 8 = -16$ as it should.

Note that the analogue of the $\partial g^I{}_J$'s in the ghost sector, $\partial h^I{}_J = \frac{1}{2} w^I \lambda_J$, are invariant under the gauge transformation $\delta w^I = \Lambda^m(\gamma_m \lambda^I)$ and are thus a priori allowed operators. However, not all $h^I{}_J$'s appear in the stress tensor (modulo normal ordering $\partial h = h^1{}_1 + h^2{}_2$ and $\partial u = h^1{}_1 - h^2{}_2$). This situation is similar to the $d = 4$ case (see the previous section) where $\hat{h}$ did not appear in $T$ but $\hat{g}$ did. The fact that not all objects which are (classically) invariant under the $\delta w^I_\alpha$ gauge symmetry appear in $T$ is perhaps not so strange, but more puzzling is the fact that we have not been able to choose the normal-ordering constants in such a way that both $N^{mn}$ and *all* $\partial h^I{}_J$'s satisfy Lorentz-covariant *and* Sp(2)-covariant OPEs. It is unclear to us whether this represents a real problem since not all $\partial h^I{}_J$'s appear in $T$ anyway.

As in the $d = 4$ case the zero-mode saturation rule for the above $d = 6$ model can be obtained by writing $T$ as a collection of $bc$-type and $\beta\gamma$-type systems all of weight one and using standard methods [16]. It is important that $w^2_+$ and $\lambda^+_2$ are non-trivially related to the $\beta'$ and $\gamma'$ of the corresponding weight one $\beta\gamma$ system. In particular, $\gamma' = \log \lambda^+_2$ so that $\delta(\gamma') = \lambda^+_2 \delta(\lambda^+_2)$ (see [15] for similar comments). One finds that the saturation rule is

$$\langle 0 | \left[ \lambda^+_2 \prod_{c=1}^{3} \theta^c_2 \right] \theta^+_1 \delta(\lambda^+_1) \theta^+_2 \delta(\lambda^+_2) \prod_{c=1}^{3} [\theta^c_1 \delta(\lambda^c_1)] \, |0\rangle \neq 0 \,, \tag{3.21}$$

which because of the $\delta$-functions is equivalent to

$$\langle 0 | (\lambda_2 \gamma^m \theta_2)(\theta_2 \gamma_m \theta_2) | \Omega \rangle \neq 0 \,, \tag{3.22}$$

where (as in $d = 4, 10$) $|\Omega\rangle = \prod_{A=1}^{5} Y^A |0\rangle$ with $Y^A = C^{AI}_\alpha \theta^\alpha_I \delta(C^{AI}_\alpha \lambda^\alpha_I)$ for certain $C^{AI}_\alpha$'s. The operators $Y^A$ each carry $(\lambda, \theta)$ charge $(-1, +1)$ and the above saturation rule is consistent with the background charges in $T$.

Note that the above saturation rule is not Sp(2)-covariant might be related to the difficulties in constructing all the $\partial h^I{}_J$'s at the quantum level.

To summarise: in this section we analysed the conformal field theory for the $d = 6$ pure-spinor superstring and showed that it has vanishing central charge and is such that the Lorentz current algebra has level one. We also obtained the zero-mode saturation rule. As in $d = 4$, many open problems remain.

## 4. The $d = 2$ pure-spinor superstring

An even simpler (albeit somewhat degenerate) case occurs in $d = 2$. We briefly consider the case of $\mathcal{N} = (2, 0)$ supersymmetry.[6] The left-moving (holomorphic) "matter" worldsheet

[6] We should point out that because of the peculiar nature of two dimensions the model and the results in this section should be taken with a grain of salt. In particular, some equations and normalisations differ from the corresponding ones in $d = 4, 6, 10$ and this may be an indication that some aspects of the model require modification.

fields are taken to be $(x^m, \theta_I, p^I)$, where $I = 1, 2$ (the spinor indexes only take one value so we do not write them explicitly). The Grassmann-odd fields $\theta_I$ are Majorana-Weyl spinors and $p^I$ are their conjugate momenta. The R-symmetry group is SO(2). We take the world-sheet ghost fields to be bosonic Majorana-Weyl spinors $\lambda_I$, satisfying the constraint

$$\delta^{IJ}\lambda_I\gamma^m\lambda_J = 0\,, \qquad I, J = 1, 2\,. \tag{4.1}$$

In the U(1) basis the pure-spinor constraint reads $\sum_I(\lambda_I^+)^2 = 0$. Assuming the $\lambda_I^+$'s are complex we can eliminate $\lambda_2^+$ and gauge-fix $w_+^2 = 0$. Using the short-hand notation $\lambda_1^+ = \lambda^+$ and $w^+ = w_1^+$, one finds that in the U(1) basis, the Lorentz-invariant quantities are:

$$N^1{}_1 = \frac{1}{2}w_+\lambda^+\,, \tag{4.2}$$

and (using a convenient normalisation)

$$\partial h = w_+\lambda^+ + \frac{\partial\lambda^+}{\lambda^+}\,. \tag{4.3}$$

From the above expressions it follows that $N^{mn}$ forms an SO(2) current algebra with level $k = -1$. Thus $L^{mn} = N^{mn} + M^{mn}$ has level $k = 1$ as required, since $M^{mn} = -\frac{1}{2}p^I\gamma^{mn}\theta_I$ has level $k = 2$. Furthermore, $\partial h$ has no singularities with $N^{mn}$, and satisfies

$$h(y)h(z) \sim \log{(y-z)}\,, \qquad \partial h(y)\lambda_I(z) \sim \frac{1}{y-z}\lambda_I(z)\,. \tag{4.4}$$

(Note that these OPEs differ slightly from the corresponding ones in $d = 4, 6, 10$.) Finally, using normal-ordering rearrangements, one finds

$$T_{N,\partial h} = \frac{1}{4}N_{mn}N^{mn} + \frac{1}{2}\left(\partial h\right)^2 = ((w_+\lambda^+) + \frac{1}{2}\frac{\partial\lambda^+}{\lambda^+})\frac{\partial\lambda^+}{\lambda^+} = -\beta'\,\partial\gamma'\,, \tag{4.5}$$

where we have defined $\beta' = -((w_+\lambda^+) + \frac{1}{2}\frac{\partial\lambda^+}{\lambda^+})$ and $\gamma' = \log\lambda^+$. Here $\beta', \gamma'$ satisfies the usual OPEs of a weight one $\beta\gamma$-system. The central charge of (4.5) is $c = +2$. Thus the total central charge vanishes since $T_{\rm mat} = -\frac{1}{2}\partial x^m\partial x_m - \delta^{IJ}p_I\partial\theta_J$ has $c = 2 - 4 = -2$.

Using the above results and $\delta(\gamma') = \lambda^+\delta(\lambda^+)$ one finds that the saturation rule becomes

$$\langle 0|\epsilon^{IJ}\theta_I^+\theta_J^+\lambda^+\delta(\lambda^+))|0\rangle \neq 0\,. \tag{4.6}$$

which is equivalent to

$$\langle 0|\epsilon^{IJ}\lambda_I\theta_J|\Omega\rangle \neq 0\,, \tag{4.7}$$

where $|\Omega\rangle = Y|0\rangle$ and $Y = C^I\theta_I\,\delta(C^I\lambda_I)$ for some $C^I$.

## 5. Relation to RNS?

In this section we discuss the relation of the new lower-dimensional pure-spinor models to compactifications of the RNS superstring. But before turning to the new models it is useful to recall what is known about the map between the $d = 10$ RNS and pure-spinor superstrings [5] (see also [12]). We should stress that the knowledge of this map is not yet at the level of a rigorous proof of the equivalence between RNS and the pure-spinor superstring. To fix notation we collect the bosonisation formulæ for the RNS ghost variables $(\beta,\gamma,b,c)$

$$\beta=\partial\xi e^{-\phi}\,,\qquad \gamma=\eta e^{\phi}\,,\qquad \xi=e^{\chi}\,,\qquad \eta=e^{-\chi}\,,\qquad c=e^{\sigma}\,,\qquad b=e^{-\sigma}\,,\tag{5.1}$$

as well as those for the RNS worldsheet fermions, $\Psi^m$ (here $a=1,\dots,5$)

$$\psi_a\equiv\Psi^a-i\Psi^{a+5}=e^{+\tau^a}\,,\qquad \psi^a\equiv\Psi^a+i\Psi^{a+5}=e^{-\tau^a}\,.\tag{5.2}$$

To relate the RNS and pure-spinor superstrings, the first step is to change variables from the RNS variables to the (GSO projected) variables introduced in [5, 18] by Berkovits. In terms of these variables, a U(5) subgroup of the (Wick-rotated) SO(10) super-Poincaré symmetry is manifest [18]. In addition to the $x^m$'s, which are left unchanged, the new variables comprise the 12 Grassmann-odd variables

$$\begin{aligned}\theta^a&=e^{\phi/2-\tau^a+\sum_b\tau^b/2}\,, &\qquad \theta^+&=e^{\sigma+\chi-3\phi/2-\sum_a\tau^a/2}\,,\\ p_a&=e^{-\phi/2+\tau^a-\sum_b\tau^b/2}\,, &\qquad p_+&=e^{-\sigma-\xi+3\phi/2+\sum_a\tau^a/2}\,,\end{aligned}\tag{5.3}$$

as well as the two Grassmann-even ones

$$s=\sigma-\frac{3}{2}\phi-\frac{1}{2}\sum_{a=1}^{5}\tau^a\,,\qquad t=-\chi+\frac{3}{2}\phi+\frac{1}{2}\sum_{a=1}^{5}\tau^a\,.\tag{5.4}$$

The next step is to add to the above variables the ten "topological" quartets ($p^{ab}$, $\theta_{ab}$, $v^{ab}$, $u_{ab}$) (here $p_{ab}=-p_{ba}$ etc.). This is a sum of ten $bc$- and ten $\beta\gamma$-type systems all with weight one and consequently each quartet has central charge $c=0$. After the addition of these quartets the field content is exactly that of the pure-spinor superstring. The Grassmann-odd variables ($\theta^+$, $\theta^a$, $\theta_{ab}$) span a 16-dimensional spinor $\theta^\alpha$ and $(\lambda^+,\lambda_{ab})=(e^s,u_{ab})$ make up the eleven components of a pure spinor, $\lambda^\alpha$.

In terms of the pure-spinor variables (and after adding the ten quartets) the RNS stress tensor becomes (suppressing the $x^m$ piece)

$$T=\partial s\partial t+\partial^2 s+\frac{1}{2}v^{ab}\partial u_{ab}-p_+\partial\theta^+-p_a\partial\theta^a-\frac{1}{2}p^{ab}\partial\theta_{ab}\,.\tag{5.5}$$

The following OPEs are non-vanishing (our conventions are as in [15])

$$\begin{aligned}t(y)\,s(z)&\sim\log(y-z)\,, &\qquad v_{ab}(y)\,u^{cd}(z)&\sim-\frac{\delta^{cd}_{ab}}{y-z}\,,\\ p_+(y)\,\theta^+(z)&\sim\frac{1}{y-z}\,, &\qquad p_a(y)\,\theta^b(z)&\sim\frac{\delta^b_a}{y-z}\,,\\ p_{ab}(y)\,\theta^{cd}(z)&\sim\frac{\delta^{cd}_{ab}}{y-z}\,.\end{aligned}\tag{5.6}$$

It is also useful to recall the discussion of the zero-mode saturation rule presented in [15] (see [15] for further details). By redefining the $s$, $t$ variables according to

$$s = \frac{1}{2}(\chi' - \phi')\,, \qquad t = \chi' + \phi'\,, \tag{5.7}$$

one finds in the $s$, $t$ sector

$$T_{s,t} = \frac{1}{2}\partial\chi'\partial\chi' + \frac{1}{2}\partial^2\chi' - \frac{1}{2}\partial\phi'\partial\phi' - \frac{1}{2}\partial^2\phi'\,, \tag{5.8}$$

which one recognises (see e.g. [19]) as a bosonised $\beta'\gamma'$-system with weight one and therefore $T_{s,t} = -\beta'\partial\gamma'$. Note that this is *not* the same as the usual RNS $\beta\gamma$-system. From this result one sees that the above stress tensor (5.5) is simply the sum of sixteen $bc$-type systems and eleven $\beta\gamma$-type systems, all of weight one. Using standard methods [16] one then finds the following zero-mode saturation rule (in the small Hilbert space with respect to the eleven $\beta\gamma$-systems)

$$\langle\epsilon_{abcde}\theta^a\theta^b\theta^c\theta^d\theta^e \left[\theta^+ \prod_{ab=1}^{10}\theta_{ab}\right]\left[\delta(\gamma')\prod_{ab=1}^{10}\delta(u_{ab})\right]\rangle \neq 0\,. \tag{5.9}$$

The relations between $s$, $u_{ab}$ and the eleven components of the pure spinor are $\lambda_{ab} = u_{ab}$ and $\lambda^+ = e^s = \gamma'^{-1/2}$. Using these relations one can write the above saturation rule as [15] (note that $\delta(\gamma') \propto (\lambda^+)^3\delta(\lambda^+)$)

$$\langle(\lambda^+)^3\epsilon_{abcde}\theta^a\theta^b\theta^c\theta^d\theta^e \left[\theta^+ \prod_{ab=1}^{10}\theta_{ab}\right][\delta(\lambda^+)\prod_{ab=1}^{10}\delta(\lambda_{ab})]\rangle \neq 0\,, \tag{5.10}$$

which was argued in [15] to be equivalent to

$$\left\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\prod_{I=1}^{11}Y_{C^I}\right\rangle \neq 0\,, \tag{5.11}$$

where $Y_{C^I} = C^I_\alpha\theta^\alpha\delta(C^I_\beta\lambda^\beta)$, and $C^I_\alpha$ are certain non-covariant constant spinors. The result (5.11) precisely coincides with the saturation rule proposed in [17] (see also [20]).

Let us now turn to the lower-dimensional pure-spinor models. We will argue that these models correspond to the compactification-independent sector of RNS compactified on Calabi-Yau manifolds.

Let us start by discussing the $d = 4$ case. Since the pure-spinor models discussed in previous sections only preserve a lower-dimensional supersymmetry one is not restricted to make the same change of variables in all dimensions. If we keep $\theta^+$ and $\theta^a$ ($a = 1, 2$) as above but instead of $\theta^a$ ($a = 3, 4, 5$) use[7] $\Upsilon^i = e^s\theta^{i+2}$ ($i = 1, \ldots, 3$) and their conjugates $r_i = e^{-s}p_{i+2}$, we find that the RNS stress tensor becomes

$$T = \partial s\partial\tilde{t} - \frac{1}{2}\partial^2 s - p_+\partial\theta^+ - \sum_{a=1}^{2}p_a\partial\theta^a - \sum_{i=1}^{3}r_i\,\partial\Upsilon^i\,, \tag{5.12}$$

[7]Note that $\Upsilon^i = ce^{-\phi}\psi^{i+2}$ in terms of the RNS variables.

where $\tilde{t} = t - \frac{3}{2}s - \frac{3}{2}\phi - \frac{3}{2}\sum_{a=1}^{2}\tau^a - \frac{1}{2}\sum_{i=1}^{3}\tau^{i+2}$ and, importantly, compared to (5.5) the coefficient in front of $\partial^2 s$ has changed. The non-vanishing OPEs between the variables are the same as in (5.6) with the replacements $t \to \tilde{t}$, $p_{i+2} \to r_i$ and $\theta^{i+2} \to \Upsilon^i$.

By redefining the $s$, $\tilde{t}$ variables according to

$$s = -\chi' + \phi' , \qquad \tilde{t} = \frac{1}{2}(\chi' + \phi') , \tag{5.13}$$

one finds the same stress tensor as in (5.8), i.e. $T_{s,\tilde{t}} = -\beta'\partial\gamma'$, the difference being that now $\gamma' = e^s$ instead of $\gamma' = e^{-2s}$. If we now add one "topological" quartet $(p^{ab}, \theta_{ab}, v^{ab}, u_{ab})$ (here $a, b = 1, 2$ and $p_{ab} = -p_{ba}$ etc.) then the field content and stress tensor becomes exactly that of the pure-spinor superstring plus a decoupled "internal" $(r_i, \Upsilon^i)$ sector. This can be seen as follows: $(\theta^+, \theta^a, \theta_{ab})$ span a four-dimensional Grassmann-odd spinor $\theta^\alpha$, and $(\lambda^+, \lambda_{ab}) = (e^s, u_{ab})$ make up the pure spinor (cf. section 2). Note that the map we constructed is really only a local map in the sense that it maps RNS into one of the two patches in (2.6). This needs to be better understood.

One might wonder what is so special about the above change of variables. One can of course also make other changes of variables (like the one used in the $d = 10$ case). The motivation for our change of variables is that it is such that four-dimensional super-Poincaré covariance is obtained at the end (since the end result agrees with the pure-spinor model of section 2).

Note that the central charge is zero for the $d = 4$ (pure-spinor) and $d = 6$ (internal $(r_i, \Upsilon^i)$) pieces separately. Thus the inconsistency which should arise if one drops the internal piece can not be seen in the central charge calculation.

Note also that the variables in the internal sector, $(r_i, \Upsilon^i)$, transforms as triplets under SU(3). Since we only have manifest $\mathcal{N} = 1$ supersymmetry in the $d = 4$ system, it should be possible to replace the internal directions with any Calabi-Yau manifold, not just flat space as above. The three complex bosonic coordinates in the internal directions, $y^i$ say, together with the $\Upsilon^i$ seem to make up a super-Calabi-Yau structure since they can be combined into the superfields $Y^i = y^i + \alpha\Upsilon^i$, where $\alpha^2 = 0$.

One can also discuss the zero-mode saturation rule along the lines of the $d = 10$ discussion. Since we are dealing with $bc$ and $\beta\gamma$ systems of weight one (in particular $\gamma' = e^s = \lambda^+$) one obtains

$$\langle [\prod_{a=1}^{2} \theta^a][\theta^+\delta(\lambda^+)\theta_{ab}\delta(\lambda_{ab})]\rangle \neq 0 , \tag{5.14}$$

which agrees with the result found in section 2. In the internal directions one finds $\langle \prod_{i=1}^{3} \Upsilon^i \rangle \neq 0$.

To discuss the $d = 6$ model one just changes the ranges of the $a$ and $i$ indexes. One finds that the RNS stress tensor becomes

$$T = \partial s \partial\tilde{t} - p_+\partial\theta^+ - \sum_{a=1}^{3} p_a \partial\theta^a - \sum_{i=1}^{2} r_i\, \partial\Upsilon^i , \tag{5.15}$$

where now $\tilde{t} = t - s - \phi - \sum_{a=1}^{3}\tau^a$. Since there is now no $\partial^2 s$ term in $T$ one finds that $\gamma' = s$.

Denoting $(\theta^+,\theta^a)$ by $\theta_2^\alpha$ and $\lambda_2^+ = e^s$ and adding the four quartets $(p_\alpha^1,\theta_1^\alpha,w_\alpha^1,\lambda_1^\alpha)$ one finds exactly the field content of the $d=6$ pure-spinor model discussed in section 3: $\theta_1^\alpha$ and $\theta_2^\alpha$ span $\theta_I^\alpha$ and $\lambda_2^+$ and $\lambda_1^\alpha$ make up the pure spinor $\lambda_I^\alpha$.

The saturation rule follows as in the $d=4,10$ cases (in particular one needs to use $\gamma' = \log\lambda_2^+$)

$$\left\langle \left[\lambda_2^+ \prod_{a=1}^{3} \theta_2^a\right]\left[\theta_2^+\delta(\lambda_2^+)\theta_1^+\delta(\lambda_1^+)\prod_{a=1}^{3}\theta_1^a\delta(\lambda_1^a)\right]\right\rangle \neq 0\,, \tag{5.16}$$

which agrees with the result in section 3.

Finally to discuss the $d=2$ case we make the change of variables to $\theta^+$ and $\theta^a$ $(a=1)$ as well as $\Upsilon^i = e^{s/2}\theta^{i+1}$ $(i=1,\ldots,4)$ and their conjugates $r_i = e^{-s/2}p_{i+1}$. (The fact that this change of variables is slightly different from the $d=4,6$ cases again highlights the peculiar nature of the $d=2$ model.) Under the change of variables the RNS stress tensor becomes exactly the same as in the $d=6$ case (5.15), except that the ranges of $a$ and $i$ are different (and the definition of $\tilde{t}$ is slightly different). If we use the notation $\theta_I = (\theta^+,\theta^a)$ and $\lambda_1 = e^s$ one finds that the field content and stress tensor are exactly the same as for the $d=2$ pure-spinor model discussed in section 4.

As in $d=6$ we have $\gamma' = s = \log\lambda^1$ so the saturation rule becomes

$$\langle[\lambda_1\theta_2][\theta_1\delta(\lambda_1)]\rangle \neq 0\,, \tag{5.17}$$

which agrees with the result in section 4.

A natural question to ask is the following. If the pure-spinor models discussed in this paper are to be thought of as compactifications of the RNS superstring it should also be possible to understand them more directly as compactifications of the $d=10$ pure-spinor superstring. Let's see how this might work for the $d=4,6$ models. If we write the $d=10$ variables in U(5) representations and decompose them under the subgroup U(2)$\times$ U(3) $\subset$ U(5) we find

$$\begin{aligned}
(\theta_{ab},\lambda_{ab}):&\ \ 10 \to (1,1)\oplus(1,3)\oplus(2,3)\,,\\
(\theta^a):&\ \ 5 \to (2,1)\oplus(1,3)\,,\\
(\theta^+,\lambda^+):&\ \ \to (1,1)\,.
\end{aligned} \tag{5.18}$$

Looking at the variables that are singlets under U(3) we see that these variables make up the U(2) field content $\lambda^+,\lambda_{ab}$ and $\theta^+,\theta^a,\theta_{ab}$ i.e. exactly the same field content as in the $d=4$ pure-spinor model. Assuming that the quartets in the representations $(1,3)$ and $(2,3)$ can be removed, we are left with (in addition to the above variables) one SU(3) triplet arising from $\theta^a$. Thus this naive counting leads to the same representation content as the $d=4$ pure-spinor model. The counting for the $d=6$ works in a similar way by interchanging the roles of U(2) and U(3) and assuming that the $(2,3)$ quartets can be removed. The U(2) singlets make up the representation content of the $d=6$ pure-spinor model and there is in addition an U(2) doublet coming from $\theta^a$. It would be interesting to make the rules for compactification more precise.

To summarise: we have presented some evidence in favour of a relation between compactifications of the RNS superstring and the lower-dimensional pure-spinor models. At this stage many of the results are heuristic and it would be nice if one could understand the relation to the RNS superstring better.

## 6. Relation to hybrid superstrings?

There exists (quantum) formulations of superstring theory compactified on the Calabi-Yau manifolds $CY_l$ ($l = 2, 3, 4$) with manifest SO$(10-2l)$ super-Poincaré symmetry in the non-compact dimensions. These formulations are referred to as hybrid superstrings [13, 14] and can be obtained via a change of variables from the RNS superstring. Such formulations exist for $d = 4$ [13], $d = 6$ [14, 21] (see also [22]) and $d = 2$ [23]. For reviews, see e.g. [24, 25, 22].

The hybrid description has been most extensively developed for $d = 4$ where vertex operators for massless [13] (see also [25]) and the first massive states [26] have been studied. Scattering amplitudes for massless modes have also been investigated at tree-level [27] and at one loop [28]. Effective actions have been studied in [29].

In $d = 4$ the hybrid formulation is obtained via a change of variables starting from the RNS superstring in bosonised form. The part of the stress tensor in the $d = 4$ hybrid superstring containing the $d = 4$ modes is (see e.g. [14])

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_a \partial\theta^a - \bar{p}_{\dot{a}}\partial\bar{\theta}^{\dot{a}} - \frac{1}{2}\partial\rho\partial\rho \tag{6.1}$$

where the chiral scalar $\rho$ has the OPE $\rho(y)\rho(z) \sim -\log(y-z)$. In the hybrid model there is also a U(1) current, whose $d = 4$ part is $J = \partial\rho$. After "twisting" $T \to T + \frac{1}{2}\partial J$ we find $T_\rho = -\frac{1}{2}\partial\rho\partial\rho + \frac{1}{2}\partial^2\rho$.

Comparing the (twisted) stress tensor in the hybrid superstring to the one in the pure-spinor superstring written in terms of $\partial h$ and $N^{mn}$ (2.14), (2.15) we see that they almost agree provided that we identify $\partial\rho = \partial h$. There is still a difference though since the pure spinor expression also contains the additional piece $T = -\frac{1}{8}N_{mn}N^{mn}$. This extra piece has $c = 0$ and is in some sense "topological", similar to the quartets which were added to the RNS superstring to obtain the pure-spinor models (see the previous section).

One can also compare the vertex operators in the pure-spinor and hybrid formalisms. In [1] we studied the integrated vertex operator for the massless modes obtained from the naive BRST operator $Q = \oint \lambda^\alpha d_\alpha$. In the Weyl basis it take the form (similar to the $d = 10$ result)

$$V = \oint \left[ \partial x^m A_m + \partial\theta^a A_a + \partial\bar{\theta}^{\dot{a}} \bar{A}_{\dot{a}} + d_a W^a + \bar{d}_{\dot{a}} \bar{W}^{\dot{a}} + \frac{1}{2} N^{mn} F_{mn} \right]. \tag{6.2}$$

Here the superfields $A_m$, $A_a$, $\bar{A}_{\dot{a}}$, $W^a$, $\bar{W}^{\dot{a}}$ and $F_{mn}$ depend on $x^m$ and $\theta^\alpha$ and can all be expressed in terms of one scalar superfield (see [1] for details). If one compares the resulting expression to the result in the hybrid superstring (see e.g. [14], section 6.1) one finds perfect agreement except for the fact that in the hybrid case the $F_{mn}N^{mn}$ piece is missing. This is of course consistent with our observation that the difference between the hybrid and pure-spinor models seems to be the addition of a $N^{mn}$, $c = 0$ sector.

Roughly, the suggested relation between the pure-spinor and hybrid models can be summarised as in figure 1.

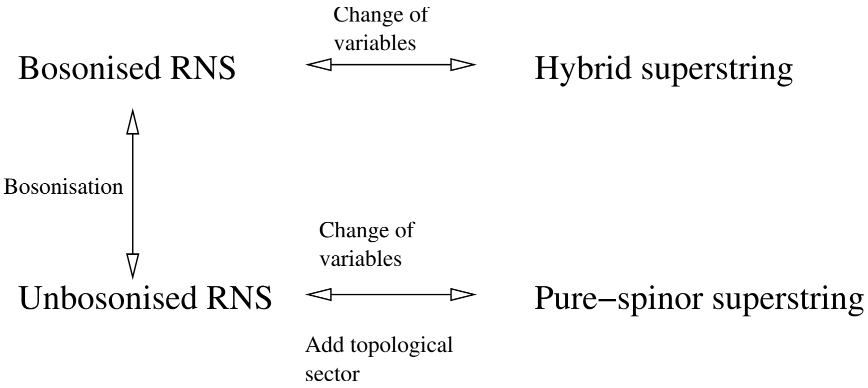


**Figure 1:** Schematic overview of suggested relation between hybrid and pure-spinor models in $d = 4$.

The role of the additional $N^{mn}$ in the pure-spinor model as compared to the hybrid model is that one gets an interpretation in terms of pure spinors. In the pure-spinor superstring one can in many cases write things in terms of $\partial h$ and $N^{mn}$, but one also has the pure spinor interpretation. Another comparison one can make is to compare the saturation rules in the two models. The saturation rule for the $d = 4$ hybrid model is (see e.g. [25])

$$0 \neq \langle e^{\rho} \rangle = \langle e^{h} \rangle \,, \tag{6.3}$$

where we used our identification of $\rho$ and $h$. On the other hand, in the pure-spinor formulation one has

$$0 \neq \langle \delta(\lambda^{+})\delta(\lambda_{ab}) \rangle = \langle e^{h+\log(\lambda^{+})} \rangle \,, \tag{6.4}$$

where we have used that (by bosonising) $\delta(\lambda^{+}) = e^{-\phi^{+}}$ where $w_{+}\lambda^{+} = -\partial\phi^{+}$ and similarly for $\delta(\lambda_{ab})$ and compared with our earlier expression for $\partial h$ (2.8). Thus we see that although both saturation rules are consistent with the background charge of $\partial h$ they are not the same. The difference might be related to a choice of large vs. small Hilbert spaces. Another interpretation might be that maybe in the hybrid model $\log(\lambda^{+})$ has somehow been absorbed into the internal Calabi-Yau piece $H_C$ (the complete saturation rule in the hybrid model is $0 \neq \langle e^{\rho+H_C} \rangle$).

One could also investigate if the $\mathcal{N} = 2, 4$ superconformal generators of the hybrid superstring can be written in pure-spinor language, as well as investigate the connection between the $\lambda$ that appears in the twistor formulation (see e.g. [25]) and the pure-spinor $\lambda$.

Next we turn to the $d = 6$ model. In the $d = 6$ hybrid formulation the part of the (untwisted) stress tensor containing the $d = 6$ modes is [14]

$$T = -\frac{1}{2}\partial x^{m}\partial x_{m} - p^{2}_{\alpha}\partial\theta^{\alpha}_{2} - \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}\partial\sigma\partial\sigma - \partial^{2}(\rho + i\sigma) \,, \tag{6.5}$$

where $\rho$, $\sigma$ have the OPEs $\rho(y)\rho(z) \sim -\log(y - z)$ and $\sigma(y)\sigma(z) \sim -\log(y - z)$. In the hybrid model there is also a U(1) current, whose $d = 6$ part is $J = -\partial(\rho + i\sigma)$.

In the $d = 6$ pure-spinor superstring the stress tensor in the ghost sector contains (see section 3) $T_{u,v} = \partial u \partial v - \partial^2 v$. By redefining $u = -\frac{1}{2}(\mu - i\nu)$ and $v = \mu + i\nu$ one finds

$$T_{\mu,\nu} = -\frac{1}{2}\partial\mu\partial\nu - \frac{1}{2}\partial\nu\partial\nu - \partial^2(\mu + i\nu)\,. \tag{6.6}$$

Comparing this to the $(\rho, \sigma)$ part of the above expression in the hybrid superstring one finds agreement provided one identifies $\mu \to \rho$ and $\nu \to \sigma$. Furthermore, using the $\partial u \partial v$ OPE one finds $\mu(y)\mu(z) \sim -\frac{1}{(y-z)^2}$ and $\nu(y)\nu(z) \sim -\frac{1}{(y-z)^2}$, so also the OPEs agree with the results in [14]. Furthermore, under the identification $e^{-\rho - i\sigma} = e^{-v} = \lambda_2^+/\lambda_1^+$. Note that in contrast to the $d = 4$ case, no twisting was needed to get agreement. We do not understand the reason for this difference.

Although the above pieces agree, in the full stress tensor there is a difference though since in the pure spinor case one has an additional $c = 0$ piece: $T = -\partial p_\alpha^1 \partial\theta_1^\alpha - \frac{1}{12}N_{mn}N^{mn} + \frac{1}{2}\partial h \partial h - \partial^2 h$ . This situation is similar to the $d = 4$ case discussed above. Note that if one writes $N^{mn} = \frac{1}{2}\kappa\gamma^{mn}\zeta$ and $\partial h = \frac{1}{2}\kappa\zeta$ in terms of two canonically conjugate Grassmann-even Weyl spinors $\kappa_\alpha$ and $\zeta^\alpha$, then $N^{mn}$ and $\partial h$ satisfy the same OPEs as in section 2. Furthermore, it can be shown that $-\frac{1}{12}N_{mn}N^{mn} + \frac{1}{2}\partial h\partial h - \partial^2 h = \kappa_\alpha \partial\zeta^\alpha$. This result makes contact with the comment in footnote 3 of [2]. One also sees explicitly that the additional $c = 0$ piece present in the pure-spinor superstring, as compared to the hybrid model, comprise four "topological" quartets $(p_\alpha^1, \theta_1^\alpha, \kappa_\alpha, \zeta^\alpha)$.

Furthermore, if one uses the solution to the pure-spinor constraint, the naive pure-spinor BRST current can be written as $\lambda_I^\alpha d_\alpha^I = \lambda_1^\alpha(d_\alpha^1 + \frac{\lambda_2^+}{\lambda_1^+}d_\alpha^2) = \lambda_1^\alpha(d_\alpha^1 + e^{-v}d_\alpha^2)$. This can be compared with the "harmonic" constraint introduced in [21]: $d_\alpha^2 - e^{-\rho - i\sigma}d_\alpha^1 \approx 0$. Using the above relation between $\rho$, $\sigma$ and $v$ one finds $e^{-\rho - i\sigma} = e^{-v} = \frac{\lambda_2^+}{\lambda_1^+}$. After trivially interchanging the labels $1 \leftrightarrow 2$ in the harmonic constraint and redefining $\theta_1$ and $d^1$ using the symmetry $\theta \to -\theta$ and $d \to -d$ one finds agreement.

To summarise: in this section we studied the relation between the pure-spinor models and the hybrid superstrings. The general structure of the relation between these models seems to be that (possibly after "twisting") the hybrid stress tensor agrees with part of the pure-spinor stress tensor. The pure-spinor stress tensor also contains an additional $c = 0$ piece. Although our results are suggestive, we do not yet have a rigorous proof of the equivalence between the hybrid and pure-spinor models.

## 7. Discussion and applications

In this paper we studied various aspects of the lower-dimensional pure-spinor superstrings introduced in [1]. Actually, referring to these models as pure-spinor superstrings is slightly premature since we have not clarified the BRST structure nor have we shown how to calculate scattering amplitudes. Pure-spinor conformal field theories might be a more appropriate name. However, we presented a tentative analysis which suggest a relation of the lower-dimensional pure-spinor models to the hybrid superstrings and to RNS compactified on Calabi-Yau manifolds. Even if these relations were to turn out not to be true, the lower-dimensional models would still be very interesting as toy models of the $d = 10$ pure-spinor model.

Despite the fact that we do not fully understand all aspects of the models, we will now discuss some possible applications of the new models. The first application is to curved backgrounds of the form $adS_n \times S^n$, and the second application is to indicate the possibility of constructing "pure-spinor M-theories" in $d = 5, 7$. In both cases we will be very brief.

### 7.1 Pure-spinor superstring theory in $adS_2 \times S^2$ and $adS_3 \times S^3$

One of the most promising aspects of the pure-spinor formalism is that it can handle backgrounds with Ramond-Ramond fields turned on. One application of obvious interest is to study string theory in $adS_5 \times S^5$ using the pure-spinor formalism. It has been shown that in this background the pure-spinor superstring sigma-model is quantisable [2]. Unfortunately the sigma model is interacting and quite complicated. Although some impressive results have been obtained (see e.g. [2, 30−33]) one would like to have a better understanding of the quantum properties of the sigma model. One approach would be to study simpler lower-dimensional models sharing several features with the $d = 10$ $adS_5 \times S^5$ model. Such models have been studied in several papers (see e.g. [34] for $adS_2 \times S^2$ models and [22, 21, 35] for $adS_3 \times S^3$ models). As we will now argue, the pure-spinor models introduced in [1] can also be studied in $adS_2 \times S^2$ and $adS_3 \times S^3$ backgrounds and thus furnishes us with examples of simplified models similar to the $adS_5 \times S^5$ pure-spinor model. Of course these models are not really new since they are very closely related to the models in [34, 22, 21]. The only difference compared with those models is that since the conformal field theories of the models discussed in this paper correspond more closely to the $d = 10$ pure-spinor model than, say, the hybrid models, the models in $adS_2 \times S^2$ and $adS_3 \times S^3$ will also correspond more closely to the $d = 10$ $adS_5 \times S^5$ pure-spinor model.

The $adS_5 \times S^5$ pure-spinor sigma model can be written

$$\int \frac{1}{2} g_{\underline{ab}} J^{\underline{a}} \bar{J}^{\underline{b}} + \frac{3}{4} \delta_{\alpha\tilde{\beta}} J^{\alpha} \bar{J}^{\hat{\beta}} + \frac{1}{4} \delta_{\alpha\tilde{\beta}} J^{\tilde{\beta}} \bar{J}^{\alpha} + \int \frac{1}{2} N_{\underline{cd}} \bar{J}^{\underline{cd}} + \frac{1}{2} \tilde{N}_{\underline{cd}} J^{\underline{cd}} + \frac{1}{4} N_{\underline{cd}} \tilde{N}^{\underline{cd}} + \int w \bar{\partial} \lambda + \tilde{w} \partial \widetilde{\lambda}\,, \tag{7.1}$$

where the currents $J^{\underline{a}}$, $J^{\alpha}$, $J^{\widetilde{\alpha}}$, $J^{\underline{cd}}$ (and the $\bar{J}$'s) belong to the Lie superalgebra $\mathrm{PSU}(2,2|4)$. In particular, $J^{\underline{cd}}$ belong to the $\mathrm{SO}(1,5) \times \mathrm{SO}(6)$ subalgebra. The terms in the first line of (7.1) are similar to the GS action constructed in [36] and the terms in the second line of (7.1) describe the couplings to the pure-spinor ghost sector.

Although we only discussed the lower-dimensional pure-spinor models in a flat supergravity background, it is straightforward to generalise to the case of a curved background space. For the cases of $adS_2 \times S^2$ and $adS_3 \times S^3$ the resulting models will take the same form as in (7.1) except that the currents now belong to the Lie superalgebras $\mathrm{PSU}(1,1|2)$ and $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$, respectively. The reason that the only difference is in the ranges of the (suitably defined) indexes is that these Lie superalgebras are very similar to $\mathrm{PSU}(2,2|4)$. For the lower-dimensional models, the first line in (7.1) is related to the GS actions constructed in [37] and [38] and the second line describes the coupling to the Lorentz currents in the ghost sector (we are assuming that there are no couplings to the Lorentz scalars).

At one loop it was shown in [34] that the terms in the first line of (7.1) are conformally invariant for the $d = 4, 6, 10$ cases. The terms in the second line of (7.1) were shown to be conformally invariant (at one loop) for the $d = 10$ case in [39]. In fact that calculation was done for general SO($2n$) and it seems that if one uses also the result for the level $k = 2 - n$ the argument works also for the $n = 2, 3$ cases, although we did not check this in detail. At higher loops one can use the argument in [33] which seems to go through *mutatis mutandis*, although again we did not check this in detail. We should also point out that conformal invariance to all orders was shown in [21] for the hybrid $adS_3 \times S^3$ model.

One can also investigate the classical flat currents which were constructed in [31] by generalising the results in [40], as well as study their quantum properties [32].

It remains to be seen if there are some calculations that can be done that are significantly simpler for the lower-dimensional models. We leave this question for future work.

### 7.2 Pure-spinor M-theory superparticle in $d = 5, 7$

In [41] the worldvolume action for the M2-brane in $d = 11$ was formulated using a pure-spinor formalism. Quantisation of the resulting model is complicated, but the superparticle limit is well defined and can be studied [41, 42]. In particular, in [42] covariant scattering amplitudes in $d = 11$ were calculated using the superparticle formalism.

A natural question to ask is if the M2-brane pure-spinor action (or its superparticle limit) can be generalised to the other dimensions were (classically) a $\kappa$-symmetric GS action exists, i.e. $d = (3), 4, 5, 7$ [43].

Before addressing this question let us briefly recall the $d = 11$ results. The pure-spinor condition is $\lambda\Gamma^M\lambda = 0$, where $\lambda^A$ is a 32-component spinor and $\Gamma^M_{AB}$ are the $32 \times 32$-dimensional gamma matrices in $d = 11$. One can decompose these results in $d = 10$ language: the spinor splits as $\lambda^A = (\lambda^\alpha, \widetilde{\lambda}_\alpha)$, and writing $M = (m, 11)$ the pure-spinor condition becomes $\lambda\gamma^m\lambda + \widetilde{\lambda}\gamma^m\widetilde{\lambda} = 0$ and $\widetilde{\lambda}\gamma^{11}\lambda = \widetilde{\lambda}\lambda = 0$. It can be explicitly checked (see [42]) that these equations imply that the pure spinor in $d = 11$ has 23 independent components. This number can be understood as arising from the 11 left-moving and 11 right-moving pure-spinor degrees of freedom of the $d = 10$ type-IIA pure-spinor superstring plus one extra mode whose interpretation was discussed in [42].

The worldline action for the $d = 11$ pure-spinor superparticle is [41, 42]

$$\int \mathrm{d}\tau \left( P_M \dot{x}^M - \frac{1}{2} P_M P^M + \dot{\theta}^A p_A + \dot{\lambda}^A w_A \right), \tag{7.2}$$

and is invariant under $\delta w_A = \Lambda_M (\Gamma^M \lambda)_A$. Next we try to generalise the $d = 11$ model to lower dimensions, focusing on the definition of the pure spinor in these dimensions.

In $d = 5$ the minimal dimension of a spinor is eight and the gamma matrices are $8 \times 8$-dimensional. We take the pure spinor, $\lambda^A$ ($A = 1, \ldots, 8$), to satisfy $\lambda\Gamma^M\lambda = 0$ ($M = 0, \ldots, 4$). Using $d = 4$ language, the spinor $\lambda^A$ can be decomposed as $\lambda^A = (\lambda^\alpha, \widetilde{\lambda}_\alpha)$. One can decompose further into U(2) representations: $\lambda^\alpha = (\lambda^+, \lambda^a, \lambda_{ab})$ as in section 2, and $\widetilde{\lambda}_\alpha = (\widetilde{\lambda}_+, \widetilde{\lambda}_a, \widetilde{\lambda}^{ab})$. The pure-spinor condition $\lambda\Gamma^M\lambda = 0$ decomposes into $\lambda\Gamma^m\lambda + \widetilde{\lambda}\Gamma^m\widetilde{\lambda} = 0$ and $\lambda\Gamma^5\widetilde{\lambda} = 0$. In the U(2) basis these equations can be written as

$$\lambda^+\lambda^a + \lambda^{ab}\widetilde{\lambda}_b = 0\,, \qquad \widetilde{\lambda}_+\widetilde{\lambda}_a + \lambda_{ab}\lambda^b = 0\,, \qquad \lambda^+\widetilde{\lambda}_+ + \lambda^a\widetilde{\lambda}_a + \frac{1}{2}\lambda_{ab}\widetilde{\lambda}^{ab} = 0\,. \tag{7.3}$$

It is easy to see that these equations eliminate three components from the eight-dimensional spinor $\lambda^A$ (e.g. $\lambda^a = -\frac{\widetilde{\lambda}^{ab}\widetilde{\lambda}_b}{\lambda^+}$ and $\widetilde{\lambda}_+ = -\frac{\lambda_{ab}\widetilde{\lambda}^{ab}}{2\lambda^+}$). The number of independent components is therefore five. Note that $5 = 2\cdot 2 + 1$ where 2 is the dimension of the pure spinor in $d = 4$, so the same counting that worked in $d = 11$ also works in $d = 5$.

The $d = 7$ case can be treated in an analogous way. We take the pure-spinor to be $\lambda_I^A$ where $A = 1, \ldots, 8$ and $I = 1, 2$. Using $d = 6$ language this spinor decomposes in the following way: $\lambda_I^A = (\lambda_I^\alpha, \widetilde{\lambda}_\alpha^I)$. The further splitting into U(3) representations is $\lambda_I^\alpha = (\lambda_I^+, \lambda_I^a)$ and $\widetilde{\lambda}_\alpha^I = (\widetilde{\lambda}_+^I, \widetilde{\lambda}_a^I)$. Using these results, the pure-spinor condition $e^{IJ}\lambda_I\Gamma^M\lambda_J = 0$ can be written

$$\epsilon^{IJ}\lambda_I^+\lambda_J^a + \frac{1}{2}\epsilon_{IJ}\epsilon^{abc}\widetilde{\lambda}_b^I\widetilde{\lambda}_c^J = 0\,, \qquad \epsilon_{IJ}\widetilde{\lambda}_+^I\widetilde{\lambda}_a^J + \frac{1}{2}\epsilon^{IJ}\epsilon_{abc}\lambda_I^b\lambda_J^c = 0\,, \qquad \lambda_I^+\widetilde{\lambda}_+^I + \lambda_I^a\widetilde{\lambda}_a^I = 0\,. \tag{7.4}$$

It can be shown that these equations eliminate five components from the sixteen-dimensional spinor $\lambda_I^A$ (e.g. $\lambda_2^a = -\frac{\lambda_2^+\lambda_1^a}{\lambda_1^+} - \epsilon^{abc}\frac{\widetilde{\lambda}_b^1\widetilde{\lambda}_c^2}{\lambda_1^+}$ as well as $\widetilde{\lambda}_+^1 = -\frac{\widetilde{\lambda}_a^1\lambda_1^a}{\lambda_1^+}$ and $\widetilde{\lambda}_+^2 = -\frac{\widetilde{\lambda}_a^2\lambda_2^a}{\lambda_2^+}$). Thus the pure-spinor contains eleven independent components. Note that $11 = 2\cdot 5 + 1$ where 5 is the dimension of the pure spinor in $d = 6$, so the same counting that worked in $d = 5, 11$ works also in $d = 7$.

Just as in $d = 11$ one can write down the tree-level saturation rule for the $d = 5, 7$ models. In $d = 2n + 1$ one finds the schematic result $\langle 0|\lambda^{2n-3}\theta^{2n-1}|\Omega\rangle \neq 0$ for $n = 2, 3, 5$. The saturation rule seems to depend only on the number of independent components of the pure spinor, since, for instance, it has the same schematic form in the $d = 7$ M-theory case as in the $d = 10$ (open) superstring case, and it has the same schematic form in the $d = 5$ M-theory case as in the $d = 6$ (open) superstring case. The action for the $d = 5, 7$ "M-theory" superparticles takes the same form as in (7.2).

One could clearly analyse these models further but we will not do so here. We also note that the $d = 3, 4$ cases appear to be subtle and will therefore not be discussed here.

**Note added:** after this work was completed the paper [44] appeared. In this work pure-spinor superstrings in $d = 4$ are also discussed. In particular, it is suggested that the $d = 4$ pure-spinor superstring, with a particular BRST operator, describes a chiral sector of superstring theory compactified on a Calabi-Yau manifold down to $d = 4$.

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## A. The U($n$) formalism

It will occasionally be useful to temporarily break SO($2n$) to U($n$) $\approx$ U(1)$\times$ SU($n$). Under this breaking pattern, the vector representation of SO($2n$) decomposes as $2n \to n \oplus \bar{n}$. The components of a SO($2n$) vector $V^m$ are related to the components of the two U($n$) representations $v^a$, $v_a$, according to $v^a = \frac{1}{2}(V^a + iV^{a+n})$ for the $n$ and $v_a = \frac{1}{2}(V^a - iV^{a+n})$ for the $\bar{n}$; here $a = 1, \dots, n$. Analogous expressions can be derived for a tensor with an arbitrary number of vector indexes. The following representations for the U($n$) components $(\gamma^a)_{\alpha\beta}$ and $(\gamma_a)_{\alpha\beta}$ of the SO($2n$) gamma matrices $\Gamma^m_{\alpha\beta}$ will be used in this paper. In $d = 2$ the 2$\times$2 matrices

$$(\gamma^1)_{\alpha\beta} = i\frac{1+\sigma_3}{2}\,, \qquad (\gamma_1)_{\alpha\beta} = -i\frac{1-\sigma_3}{2}\,. \tag{A.1}$$

are symmetric. In $d = 4$ the 4$\times$4 matrices

$$\begin{aligned} (\gamma^1)_{\alpha\beta} &= i\frac{1+\sigma_3}{2} \otimes \sigma_1\,, & (\gamma_1)_{\alpha\beta} &= -i\frac{1-\sigma_3}{2} \otimes \sigma_1\,, \\ (\gamma^2)_{\alpha\beta} &= -i\sigma_1 \otimes \frac{1+\sigma_3}{2}\,, & (\gamma_2)_{\alpha\beta} &= -i\sigma_1 \otimes \frac{1-\sigma_3}{2}\,. \end{aligned} \tag{A.2}$$

are symmetric. Finally, in $d = 6$ the 8$\times$8 matrices

$$\begin{aligned} (\gamma^1)_{\alpha\beta} &= i\frac{1+\sigma_3}{2} \otimes \sigma_1 \otimes \sigma_2\,, & (\gamma_1)_{\alpha\beta} &= -i\frac{1-\sigma_3}{2} \otimes \sigma_1 \otimes \sigma_2\,, \\ (\gamma^2)_{\alpha\beta} &= -i\sigma_1 \otimes \frac{1+\sigma_3}{2} \otimes \sigma_2\,, & (\gamma_2)_{\alpha\beta} &= -i\sigma_1 \otimes \frac{1-\sigma_3}{2} \otimes \sigma_2\,, \\ (\gamma^3)_{\alpha\beta} &= -i\sigma_1 \otimes \sigma_2 \otimes \frac{1+\sigma_3}{2}\,, & (\gamma_3)_{\alpha\beta} &= -i\sigma_1 \otimes \sigma_2 \otimes \frac{1-\sigma_3}{2}\,. \end{aligned} \tag{A.3}$$

are antisymmetric.

Indexes are raised and lowered with $\mathcal{C}^{\alpha\beta} = \sigma_2$ ($d = 2$), $\mathcal{C}^{\alpha\beta} = \sigma_2 \otimes \sigma_1$ ($d = 4$) and $\mathcal{C}^{\alpha\beta} = \sigma_2 \otimes \sigma_1 \otimes \sigma_2$ ($d = 6$) and its inverse $\mathcal{C}_{\alpha\beta}$, according to the rule $T^{\alpha\beta} = \mathcal{C}^{\alpha\delta} T_{\delta\rho} \mathcal{C}^{\rho\beta}$.

The above matrices satisfy $\{\gamma^a, \gamma_b\} = \delta^a_b$. From this result it follows that the corresponding $\Gamma^m$'s satisfy $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$.

In $d = 6$ (as in $d = 10$) the restriction of $\Gamma^m_{\alpha\beta}$ to the Weyl subspace (action on Weyl spinors) will be denoted by $\gamma^m_{\alpha\beta}$.

A spinor of SO($2n$) is conveniently represented as the direct product of $n$ SO(2) spinors. Denoting the SO(2) spinor $\binom{1}{0}$ by $+$ and $\binom{0}{1}$ by $-$, SO($2n$) spinors are naturally labelled by a composite index $(\pm, \dots, \pm)$, where all possible choices are allowed.

The above $\gamma$ matrices act on this basis in the natural way. Our conventions for the chirality matrix are such that spinors with an odd (even) number of $+$'s are Weyl (anti-Weyl) spinors. The difference between the number of $+$'s and $-$'s divided by 2 is the U(1) quantum number.

The following notation is used for SU($d$) components of a spinor $\lambda^\alpha$. In $d = 2$ the $+$ (Weyl) component is denoted $\lambda^+$. In $d = 4$ the $++$ component is denoted $\lambda^+$, the components with one $+$ are denoted $\lambda^a$ and the $--$ component is denoted by $\lambda_{ab} = -\lambda_{ba}$ ($a, b = 1, 2$). We also use the notation $\bar{\lambda}^{\dot{a}}$ for $\{\lambda^+, \frac{1}{2}\epsilon^{ab}\lambda_{ab}\}$ . Note that $\lambda^a$ is a Weyl-spinor, whereas $\bar{\lambda}^{\dot{a}}$ is an anti-Weyl spinor. In $d = 6$ the $+++$ component is denoted $\lambda^+$, and the components with one $+$ are denoted $\lambda^a$ ($a = 1, 2, 3$). These components span a Weyl-spinor.

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